Warner. Foundations of Differentiable Manifolds and Lie Groups 2017 Fall

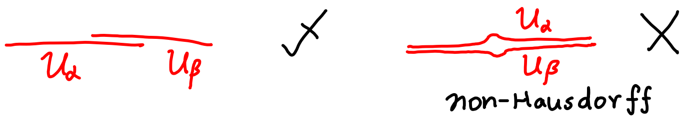
[Skip Ch. 4 & 5]

Chapter 1. Manifolds

Def. Manifold Hausdorff; 2^{nd} countable $M = \bigcup \mathcal{U}_\alpha$

$$\mathcal{U}_\alpha \xrightarrow[\simeq]{\varphi_\alpha} B^n \subset \mathbb{R}^n \quad \text{s.t.} \quad \varphi_\alpha \circ \varphi_\beta^{-1} \in C^\infty$$

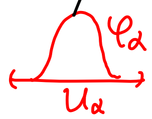
- Hausdorff



- 2^{nd} countable (topo. has a countable base)
  i.e. only need to use countably many charts $\mathcal{U}_\alpha$'s.

$\Rightarrow \exists$ partition of unity.

$$M = \bigcup_{\alpha}^{\text{countable}} \mathcal{U}_\alpha ,$$



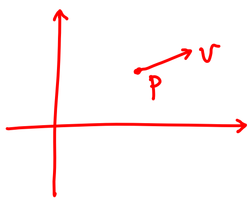
$$\overset{\text{(loc. finite)}}{\sum_\alpha \varphi_\alpha = 1}$$

($\Rightarrow \exists$ Riemannian metric, etc)

§ Tangent vectors

(i) $p \in M = \mathbb{R}^d \Rightarrow T_pM = \mathbb{R}^d$

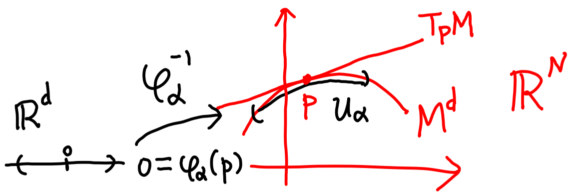
same $\forall$ p



(ii) $p \in M^d \subset \mathbb{R}^N$

$T_pM \leqslant \mathbb{R}^N$

via $\varphi_\alpha^{-1} : \mathbb{R}^d \xrightarrow[\sim]{\text{loc.}} M \subset \mathbb{R}^N$



$d(\varphi_\alpha^{-1})(0) : \mathbb{R}^d \longrightarrow \mathbb{R}^N \rightsquigarrow \text{Image} = T_pM$

(iii) Intrinsic def^{n}.

$v \in T_pM \longleftrightarrow$ differentiate (germ of) functions at p along "direction" v.

$\mathcal{O}_p \triangleq \{$ germs of fu. $f(x)$ around $p \in M \}$ comm. alg /$\mathbb{R}$

$\triangledown\!\!/$ $\qquad\qquad\qquad\qquad (f_1 + f_2 \,,\ f_1 \cdot f_2)$

$m_p \ni f(x)$ w/ $f(p) = 0$

Def: $v \in T_pM \overset{\Delta}{\Longleftrightarrow} v : \mathcal{O}_p \longrightarrow \mathbb{R}$ derivation

$$v(fg) = f(p)v(g) + g(p)v(f)$$

$$\overset{\text{lemma}}{\Longleftrightarrow} \quad v \in \left(\frac{m_p}{m_p^2}\right)^*$$

[$\Rightarrow$] $\quad f(p) = 0 = g(p) \Rightarrow v(fg) = 0$ (at p)

[$\Leftarrow$] $\quad \varphi \in (m_p/m_p^2)^* \rightsquigarrow v(f) \triangleq \varphi(f - f(p))$

$$v(fg) = \varphi(fg - f(p)g(p))$$
$$= \varphi(\underbrace{(f-f(p))(g-g(p))}_{\in m_p^2 \to 0}) + f(p)\underbrace{\varphi(g-g(p))}_{v(g)} + g(p)\underbrace{\varphi(f-f(p))}_{v(f)}$$

- $\dim T_pM = \dim M$ ($\because$ Taylor series)
- In local coord., $T_pM = \mathbb{R}\langle \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d} \rangle$

§ Differential

$$\begin{array}{ccc} M & \xrightarrow{\psi} & N \\ \psi & & \psi \\ p & & \psi(p) \end{array} \xrightarrow{g} \mathbb{R} \quad \rightsquigarrow \quad T_pM \xrightarrow{\psi_* = d\psi(p)} T_{\psi(p)}N$$

$$v \longmapsto \psi_* v$$

$$\psi_* v(g) := v(g \circ \psi)$$

- $d(\varphi \circ \psi) = d\varphi \circ d\psi$

- In bundle language, $\quad TM = \coprod_{p \in M} T_pM \xrightarrow{\pi} M$

$$d\psi \in \Gamma(M, T_M^* \otimes \psi^* T_N)$$

§ Higher order tangents / Jet bundle.

$$0 \to m_p \to \mathcal{O}_p \xrightarrow{ev} \mathbb{R} \to 0 \qquad \left(T_pM = \left(\frac{m_p}{m_p^2}\right)^*\right)$$

split $\Downarrow$ const map

k^{th}-jet $\quad \mathcal{J}_{k,p} := \left(\frac{\mathcal{O}_p}{m_p^{k+1}}\right)^* = \mathbb{R} \oplus \left(\frac{m_p}{m_p^{k+1}}\right)^*$

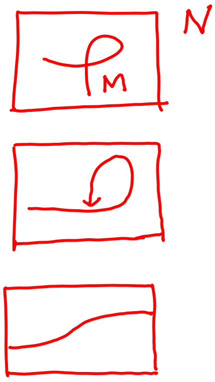
- $\mathcal{J}_{1,p} = \mathbb{R} \oplus T_pM \qquad \underbrace{\mathrm{Sym}^{k+1}}\, T_pM$ ($\because$ Taylor Series)
- $0 \to \mathcal{J}_{k-1} \to \mathcal{J}_k \to \left(\frac{m^k}{m^{k+1}}\right)^* \to 0$

§ Submanifolds (parametrized) $\psi : M \longrightarrow N$

1) immension if $T_pM \xhookrightarrow[1-1]{d\psi_p} T_{\psi(p)}N \quad \forall p$

2) submanifold if $+ \quad \psi \quad 1-1$

3) embedding if $+ \quad M \xrightarrow[\text{homeo.}]{\psi} \psi(M)$



- $\begin{array}{l} \text{submfd } M_1 \\ \text{diffeo} \downarrow \quad \circlearrowleft \\ \text{submfd } M_2 \end{array}$ with $M_1 \xhookrightarrow{\psi_1} N$, $M_2 \xhookrightarrow{\psi_2} N$ $\Longrightarrow$ equivalent/same (namely, unparametrized)

§ Implicit Function Theorem

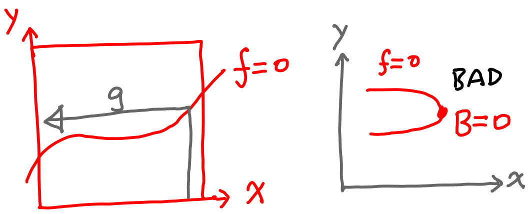
Thm. $f : \mathbb{R}^m \times \mathbb{R}^n \xrightarrow{C^\infty} \mathbb{R}^n$ w/ $f(0) = 0$,

B non-singular, where $df(0) = \begin{pmatrix} A_{mn} \\ B_{nn} \end{pmatrix}_{(m+n)\times n}$

$\Rightarrow$ locally, $\exists\, C^\infty$ $g : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

s.t. $f(x,y) = 0 \iff y = g(x)$

i.e. $\{ f = 0 \} = \text{Graph}(g)$



Cor. $\begin{array}{ccc} M & \xrightarrow{\psi} & N \\ \cup\!\!| & & \Psi \\ P := \psi^{-1}(y) & \longrightarrow & y \end{array}$ $\quad \forall x \in P$, $T_xM \underset{\text{onto}}{\xrightarrow{\psi_x}} T_yN$

$\Rightarrow P \xhookrightarrow[\text{submfd}]{\text{emb.}} M$

$\dim P = \dim M - \dim N$

§ Vector fields $X, Y \in \Gamma(M, T_M)$

Claim: $[X,Y] = XY - YX$ is v.f.

$$\begin{aligned}
&(XY - YX)(fg) \\
&= X\big((Yf)g + f(Yg)\big) - Y\big((Xf)g + f(Xg)\big) \\
&= (XYf)g + \cancel{(Yf)(Xg)} + \cancel{(Xf)(Yg)} + f(XYg) \\
&\quad - \big[(YXf)g + \cancel{(Xf)(Yg)} + \cancel{(Yf)(Xg)} + f(YXg)\big] \\
&= (XYf - YXf)g + f(XYg - YXg)
\end{aligned}$$

Equivalently, $\left[a(x,y)\frac{\partial}{\partial x}, b(x,y)\frac{\partial}{\partial y}\right] f$

$$\begin{aligned}
&= a\,\partial_x(b f_{,y}) - b\,\partial_y(a f_{,x}) \\
&= a b_{,x} f_{,y} - b a_{,y} f_{,x} \leftarrow 1^{st} \text{ diff.} \\
&\quad + ab\, \cancel{f_{,yx}} - ba\, \cancel{f_{,xy}} \quad \left(\because \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}\right)
\end{aligned}$$

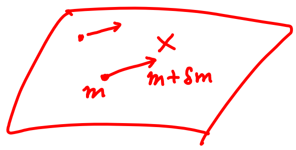
- $(\Gamma(M, T_M), [\ \ ])$ is Lie alg.

(i.e. $[X, Y] = -[Y, X]$

$[[X\ Y]\ Z] + [[Y\ Z]\ X] + [[Z\ X]\ Y] = 0$ Jacobi identity

- $\Gamma(M, T_M) = T_{Id.}(\mathrm{Diff}(M))$

  $=: \mathrm{Lie\,Diff}(M)$

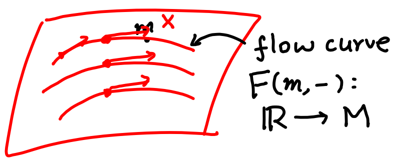


Lie alg. of an "∞ dim" Lie group.

Say M compact, then $\forall\ X \in \Gamma(M, T_M)$,

$\Rightarrow \exists$ "flow" $F: M \times \mathbb{R} \to M$

(write $f_t = F(-,t) \in \mathrm{Diff}(M)$)



s.t. $f_0 = id_M$, $\left.\frac{df_t}{dt}\right|_{t=0} = X$, $f_{t_1+t_2} = f_{t_1} \circ f_{t_2}$

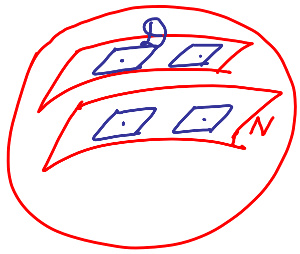
Write $f_t =: e^{tX}: M \to M$ (flow along X for time t)

(If M noncpt, flow from m may not exist $\forall t$, dep. on m).

§ Frobenius Theorem.

$\mathcal{D}$ distribution on M

$\overset{\Delta}{\Longleftrightarrow}$ $\mathcal{D}^k \leq T_M$, k-dim. subbundle



<u>Theorem</u>. $\mathcal{D}$ forms a foliation

(def: $\forall\, m \in M$ $\exists$ submfd. $N \ni x$ s.t. $T_y N = \mathcal{D}_y$ $\forall\, y \in N$)

$\Longleftrightarrow$ $\mathcal{D}$ is integrable / involutive

(def: $\forall\, X, Y \in \Gamma(M, \mathcal{D}) \subseteq \Gamma(M, T_M) \Rightarrow [X, Y] \in \Gamma(M, \mathcal{D})$)

(In particular, integrability is automatic if $\operatorname{rank} \mathcal{D} = 1$, namely, solving O.D.E. $\Rightarrow$ integral curves.)

Proof: [$\Rightarrow$] $[\ ]$ of v.f. on submfd $N \subset M$ does not care about M.

i.e. $[b(x,y,z)\frac{\partial}{\partial y}, c(x,y,z)\frac{\partial}{\partial z}] \in \mathrm{Span}\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$

[$\Leftarrow$] Local question, WLOG $\mathcal{D}^2 \leq T\mathbb{R}^3$.

$\nexists$ coord. x, y, z near 0, s.t. $\mathcal{D}_p = \mathbb{R}\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle \ \forall\ p \sim 0$.

Choose v.f. X, Y span $\mathcal{D}$ (everything is local around 0)

Choose coord. s.t. $X = \frac{\partial}{\partial x}$, write $Y = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$

Can assume $a \equiv 0$.

$$[X, Y] = b_x \frac{\partial}{\partial y} + c_x \frac{\partial}{\partial z} \overset{\text{assumption}}{\in} \mathcal{D} = \langle X, Y\rangle$$
$$= \lambda Y$$

$$[X, e^{\varphi}Y] = X(\varphi)e^{\varphi}Y + e^{\varphi}[X,Y]$$
$$= (X(\varphi) + \lambda)e^{\varphi}Y$$

Solve for φ s.t. $X(\varphi) + \lambda = 0$, i.e. $\varphi = -\int_0^x \lambda\, dt$

Replace $Y \mapsto e^{\varphi}Y \Rightarrow 0 = [X, Y]$

$\Rightarrow$ v.f. Y is tangent to coord. $\langle y, z\rangle$-plane, P

Coord. change in $\langle y, z\rangle$-plane s.t. $Y = \frac{\partial}{\partial y}$ (leaving x alone)

i.e. $Y = b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$ s.t. $Y|_P = \frac{\partial}{\partial y}$

i.e. $b(0, y, z) = 1$ & $c(0, y, z) = 0$

$$0 = [X, Y] = b_x \frac{\partial}{\partial y} + c_x \frac{\partial}{\partial z}$$
$$\Longrightarrow \quad b \equiv 1 \quad \& \quad c \equiv 0$$

i.e. $\exists$ loc. coord. x, y, z s.t. $X = \frac{\partial}{\partial x}$ & $Y = \frac{\partial}{\partial y}$. Q.E.D.

Chapter 2. Tensors & Differential Forms.

§ Linear algebra $\quad V \quad (\simeq \mathbb{R}^n)$

$\rightsquigarrow$ tensor alg. $\quad \mathcal{T}(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k}$

exterior alg. $\quad \Lambda^{\bullet}(V) := \mathcal{T}(V) \big/$ 2-sided ideal generated by $v \otimes v$'s.

$\subset \mathcal{T}(V) \qquad$ i.e. $u \wedge v = -v \wedge u$

base $\quad \underbrace{1}_{\Lambda^0}, \underbrace{e_1, e_2, e_3}_{\Lambda^1}, \underbrace{e_1\wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3}_{\Lambda^2}, \underbrace{e_1 \wedge e_2 \wedge e_3}_{\Lambda^3}$

$$\dim \Lambda^k(V) = \binom{n}{k} \qquad (\Rightarrow 0 \le k \le n)$$

- $u \in V \setminus 0 \rightsquigarrow \varepsilon(u) : \Lambda^k(V) \xrightarrow{u\wedge} \Lambda^{k+1}(V)$

$\overset{\text{transpose}}{\rightsquigarrow} \; i(u) : \Lambda^{k+1}(V^*) \xrightarrow{u \lrcorner} \Lambda^k(V^*)$

§ Differential forms. and d

M^n mfd. $\rightsquigarrow$

$$\begin{array}{ccccc} \mathcal{T}(T_M) & \supset & \mathcal{T}(T_x M), & & \Lambda^{\bullet} T_M^* \quad \text{bdl.} \\ \downarrow & & \downarrow & & \downarrow \\ M & \ni & x & & M \end{array}$$

$\Omega^k(M) := \Gamma(M, \Lambda^k T_M^*)$ differential form deg k.

$\Omega^0(M) = C^\infty(M)$

- $$\begin{array}{ccccc} \Gamma(M, T_M) & \otimes & C^\infty(M) & \longrightarrow & C^\infty(M) \\ X & & f & & X(f) \end{array}$$

$\overset{\text{dual}}{\rightsquigarrow} \; C^\infty(M) = \Omega^0(M) \xrightarrow{d} \Omega^1(M) = \Gamma(M, T_M^*)$

$f \longmapsto df \quad$ by $\quad df(X) \triangleq X(f)$

e.g. $f : \mathbb{R}^2_{x_1,x_2} \longrightarrow \mathbb{R} \Rightarrow df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2$

(i.e. $X(f) = X(x_1)\frac{\partial f}{\partial x_1} + X(x_2)\frac{\partial f}{\partial x_2}$ change rule)

"d again" $\rightsquigarrow$ $(f_{x_1x_1} dx_1 + \underline{f_{x_1x_2}} dx_2) \otimes dx_1$

$+ (\overline{f_{x_2x_1}} dx_1 + f_{x_2x_2} dx_2) \otimes dx_2$

Fubini

$\Rightarrow = 0 \in \Omega^2(M)$; $dx_1 \wedge dx_1 = 0$, $dx_1 \wedge dx_2 = -dx_2 \wedge dx_1$

In general, $\begin{cases} d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M) \\ d^2 = 0 \\ d(\varphi \wedge \eta) = d\varphi \wedge \eta \pm \varphi \wedge d\eta \end{cases}$ exterior derivative

- $M \xrightarrow{\psi} N \rightsquigarrow$ pullback

(i.e. change rule)

$$\begin{array}{ccc} \Omega^k(N) & \xrightarrow{\psi^*} & \Omega^k(M) \\ d \downarrow & \circlearrowleft & \downarrow d \\ \Omega^{k+1}(N) & \xrightarrow{\psi^*} & \Omega^{k+1}(M) \end{array}$$

§ Lie Derivative. $\mathrm{Diff}(M) \curvearrowright M$ nonlinear action

(M cpt, say)

$\rightsquigarrow \mathrm{Diff}(M) \curvearrowright \Gamma(M, T_M^{\otimes r} \otimes T_M^{*\otimes s})$

linear action/representat$^{\underline{n}}$.

$\rightsquigarrow \underbrace{\mathrm{LieDiff}(M)}_{\Gamma(M, T_M)} \curvearrowright \Gamma(\text{———} \text{ " } \text{———})$

$X \longmapsto \mathcal{L}_X$ Lie derivative

($v \in T_pM \rightsquigarrow$ differentiate $f: M \to \mathbb{R}$, $f \in \Gamma(M, \underline{\mathbb{R}})$

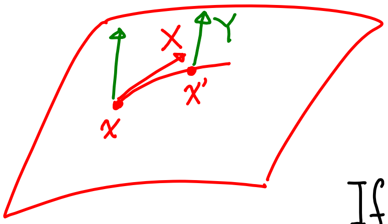
But need $v \in \Gamma(M, T_M)$ to differentiate $\Gamma(M, \otimes T_M^*)$,

$\because$ need flow to identify T_pM and nearby T_qM's, $q \sim p$.)

$$X \in \Gamma(T_M) \xrightarrow[\text{flow}]{\text{generate}} F : M \times \mathbb{R} \longrightarrow M$$

$$F_t = F(-,t) \in \mathrm{Diff}(M)$$

$$\text{“} = e^{tX} \text{”}$$

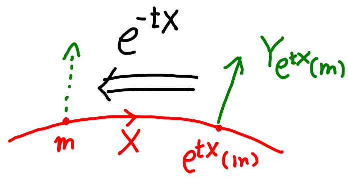


If $F_{\delta t}(x) = x' \Rightarrow T_xM \underset{dF_{-\delta t}}{\overset{dF_{\delta t}}{\rightleftarrows}} T_{x'}M$

$$\mathcal{L}_X Y = \lim_{\delta t \to 0} \frac{dF_{-\delta t}(Y(x')) - Y(x)}{\delta t}$$

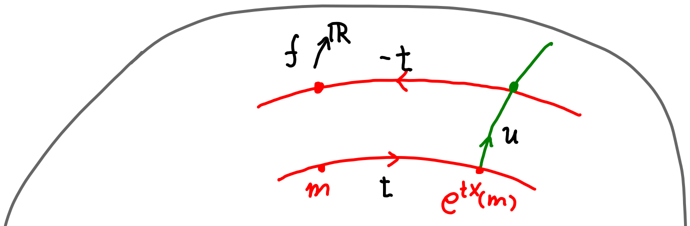
$$\mathcal{L}_X \overset{T_M^* \,\ni}{\varphi} = \lim_{\delta t \to 0} \frac{F^*_{\delta t}(\varphi(x')) - \varphi(x)}{\delta t}$$

- On $C^\infty(M)$, $\mathcal{L}_X f = X(f)$
- On $\Gamma(T_M)$, $\mathcal{L}_X Y = [X, Y]$



Pf: $(\mathcal{L}_X Y)(f) = \frac{d}{dt}\Big|_{t=0} \big((e^{-tX})_*(Y_{e^{tX}(m)})\big)(f)$

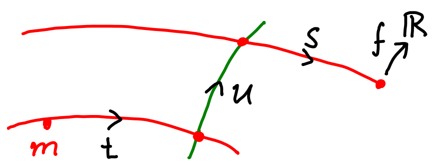
$$= \frac{d}{dt}\Big|_{t=0} \underbrace{Y_{e^{tX}(m)}(f \circ e^{-tX})}_{\frac{\partial}{\partial u}\big|_{(t,0)} H(t,u)}$$



$H(t,u) \triangleq f(e^{-tX} \circ e^{uY} \circ e^{tX}(m))$

Fubini

$$= \underbrace{\frac{\partial}{\partial u} \frac{\partial}{\partial t}\Big|_{(0,0)} \underbrace{H(t,u)}_{K(t,u,-t)}}_{\frac{\partial K}{\partial t} - \frac{\partial K}{\partial s}}$$



$K(t,u,s) \triangleq f(e^{sX} \circ e^{uY} \circ e^{tX}(m))$

$$\frac{\partial}{\partial t} \frac{\partial}{\partial u}\Big|_{(000)} K = \frac{\partial}{\partial t}\Big|_{(000)} \underbrace{\left(\frac{\partial K}{\partial u}\Big|_{(t00)} \right)}_{Y_{e^{tX}(m)}(f)} = X(Yf)$$

$$\frac{\partial}{\partial s} \frac{\partial}{\partial u}\Big|_{(000)} K = \dots = X(Yf)$$

$$\Rightarrow \mathcal{L}_X Y = XY - YX$$

Q.E.D.

- $\mathcal{L}_X$, d commute on $\Omega^*(M)$

  ($\because$ 1) d commutes w/ F^* for $F: M \to N$ (change rule)
  2) $\mathcal{L}_X \varphi = \lim_{t\to 0} \frac{d}{dt} F_t^*(\varphi)$ w/ $F_t: M \to M$ flow of X

- Cartan formula : On $\Omega^*(M)$

$$\mathcal{L}_X = d\,\iota_X + \iota_X\, d = \{ d , \iota_X \}$$

$$\mathcal{L}_{\underbrace{b^j(x)\frac{\partial}{\partial x^j}}_{X}}(a_i(x)dx^i) = (\mathcal{L}_X a_i)dx^i + a_i \mathcal{L}_X(dx^i) \quad \text{(derivat}^{\text{n}}\text{)}$$
$$= X(a_i)\, dx^i + a_i\, d(X(x^i)) \quad (\because [\mathcal{L}_X, d]=0)$$
$$= b^j a_{i,j}\, dx^i + a_i\, d(b^j \delta^i_j)$$
$$= \text{——}\,''\,\text{——} + a_i\, b^i{}_{,k}\, dx^k$$

$$d\,\iota_{b^j\partial_j}(a_i dx^i) = d(b^j a_i \delta^i_j) = \cancel{a_{i,k} b^i dx^k} + a_i b^i{}_{,k} dx^k$$

$$\iota_{b^j\partial_j} d(a_i dx^i) = \iota_{b^j\partial_j}(a_{i,k} dx^k \wedge dx^i) = b^j a_{i,j} dx^i - \cancel{b^j a_{j,k} dx^k}$$

§ Differential Ideals

$\mathcal{D} \leqslant T_M \rightsquigarrow \mathcal{I} := \{ \varphi \in \Omega^\bullet(M) : \varphi|_{\mathcal{D}} = 0 \}$

distribution

- $\mathcal{I} \trianglelefteq \Omega^\bullet(M)$ (ideal)
- loc., $\exists\ \omega_1, \dots, \omega_{n-p} \in \Omega^1$ ($n-p = \operatorname{codim}\mathcal{D}$)
  - linearly indep at every point
  - $\mathcal{I}$ = ideal gen. by ω_i's.

Converse ✓ (i.e. $\mathcal{D} \leftsquigarrow \mathcal{I}$)

Prop: $\mathcal{D}$ integrable $\iff$ $d\mathcal{I} \subset \mathcal{I}$ (differential ideal)

(reason : $\omega[X,Y] = -d\omega(X,Y) + X\omega(Y) - Y\omega(X)$.

Chapter 3. Lie groups.

Def. G Lie group $\overset{\Delta}{\Leftrightarrow}$ group + manifold
(compatible)

$$\begin{tikzcd} & TG \arrow[d] \\ G \arrow[r, bend left, "\text{Left translat}^{n}"] \arrow[ur, bend left] & G \end{tikzcd} \quad \rightsquigarrow \quad G \curvearrowright \Gamma(G, TG)$$

$$\begin{array}{c} \Gamma(G,TG) \\ \cup\!| \\ \mathfrak{g} := T_eG \cong \Gamma(G,TG)^{G_L} \end{array}$$
$\leftarrow$ left inv. vector fields

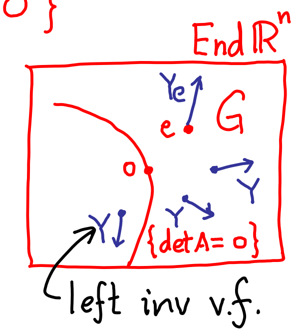
- [left inv. v.f. , left inv. v.f] is left inv.

$\Rightarrow \mathfrak{g}$ Lie algebra $\left(\begin{array}{l} \text{i.e. } [-,-] : \Lambda^2 \mathfrak{g} \longrightarrow \mathfrak{g} \text{ s.t. Jacobi identity:} \\ [[X\,Y]\,Z] + [[Y\,Z]\,X] + [[Z\,X]\,Y] = 0 \end{array} \right)$.

Eg. $G = GL(n, \mathbb{R}) \subset \mathfrak{gl}(n,\mathbb{R}) = \mathrm{End}(\mathbb{R}^n) \simeq \mathbb{R}^{n^2}$

complement of hypersurface $\{\det(A) = 0\}$

So $\Gamma(G, T_G)^{G_L} \simeq T_eG \simeq \mathrm{End}(\mathbb{R}^n)$



Claim: $\underbrace{[X, Y]_e}_{\text{as Lie alg. of } G} = \underbrace{X_e Y_e - Y_e X_e}_{\text{as matrix}}$

Proof: $x_{ij} : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}$, $x_{ij}(\sigma) \overset{\Delta}{=} a_{ij}$ if $\sigma = (a_{kl})_{n \times n}$

$$[X,Y]_e(x_{ij}) = X_e\big(\underline{Y(x_{ij})}\big) - Y_e\big(X(x_{ij})\big)$$

$$Y(x_{ij})(\sigma) \xlongequal{\because \text{left inv.v.f.}} dl_\sigma(Y_e)(x_{ij}) = Y_e(\underline{x_{ij} \circ l_\sigma})$$

$$(x_{ij} \circ l_\sigma)(\tau) = x_{ij}(\sigma\tau) \xlongequal[\text{multi.}]{\text{matrix}} x_{ik}(\sigma)\, x_{kj}(\tau)$$

$$\Rightarrow \underbrace{Y_e(x_{ij} \circ l_\sigma)}_{Y(x_{ij})(\sigma)} = x_{ik}(\sigma)\, Y(x_{kj}(\tau)) = x_{ik}(\sigma)(Y_e)_{kj}$$

$$\Rightarrow X_e(Y(x_{ij})) = (X_e)_{ik}(Y_e)_{kj} \Rightarrow [X,Y]_e = X_eY_e - Y_eX_e$$

Q.E.D.

$\varphi : G \longrightarrow H$ (smooth) homomorphism

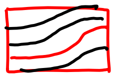
$\Rightarrow$ left inv. v.f. $\xmapsto{\varphi_*}$ left inv. v.f. (on image)

$\Rightarrow$ $\varphi_* : \mathfrak{g} \longrightarrow \mathfrak{h}$ Lie alg homomorphism

Thm: $G \overset{\varphi}{\underset{\psi}{\rightrightarrows}} H$ homomorphisms

$\varphi_* = \psi_* : \mathfrak{g} \longrightarrow \mathfrak{h}$

G connected $\Rightarrow \varphi = \psi$.

Pf: (1) $\varphi : G \to H \Rightarrow \mathrm{Graph}(\varphi) \subseteq G \times H$ 

is a leave of a foliation (via H-transl).

(2) This distribut^{n} only use left inv. forms,

thus control by $\mathfrak{g}$ & $\mathfrak{h}$.

Prop. $e \in U \underset{\text{open}}{\subset} G$ connected Lie group

$\Rightarrow$ $G = \bigcup_{n=1}^{\infty} U^n$ (i.e. U generates G)

$U^n := \{ g_1 \cdot \dots \cdot g_n \in G \mid g_i \in U \}$

Pf: WLOG, $U = U^{-1}$ $\rightsquigarrow$ $\bigcup^{\infty} U^n \leq G$ open. subgp.

$= G$. $\Leftarrow$ closed

Theorem. $\tilde{\mathfrak{h}} \leq \mathfrak{g} = \mathrm{Lie}\, G$

$\Rightarrow \exists !$ conn. $H \leq G$ s.t. $\mathfrak{h} = \tilde{\mathfrak{h}}$

(Pf: $\tilde{\mathfrak{h}} \rightsquigarrow$ distribution; Lie subalg $\Rightarrow$ integ.).

Theorem $H \leq G$

embedding $\Longleftrightarrow$ closed

§ Covering

$$\begin{array}{c} \tilde{G} \\ \downarrow \\ G \end{array} \quad \pi_1 = 0 \qquad \text{covering, Lie gp. homo.}$$

"Meta Thm": $\tilde{G} \longleftrightarrow \mathfrak{g}$

§ Exponential Map

$$X \in \mathfrak{g} \rightsquigarrow \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathfrak{g} \\ 1 & \longmapsto & X \end{array} \xrightarrow[\pi_1\mathbb{R}=0]{\int} \begin{array}{ccc} \mathbb{R} & \longrightarrow & G \\ 1 & \longmapsto & \exp(X) \end{array}$$

1-parameter subgp.

$$\rightsquigarrow \exp : \mathfrak{g} \longrightarrow G$$

Properties: 1) $d(\exp)(0) : T_0\mathfrak{g} \longrightarrow T_e G$
(Exercise)

$$\begin{array}{ccccc} d(\exp)(0) & & T_0\mathfrak{g} & & T_e G \\ \| & & \| & & \| \\ \text{id.} & & \mathfrak{g} & & \mathfrak{g} \end{array}$$

2)
$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\varphi_*} & \mathfrak{g} \\ {\scriptstyle \exp}\downarrow & \circlearrowleft & \downarrow \\ H & \xrightarrow{\varphi} & G \end{array}$$

3) $\exp : \mathfrak{gl}(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$

$$A_{n\times n} \longmapsto e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

§ "$C^0 \xRightarrow{\text{Group}} C^\infty$"

<u>Thm</u>. $\varphi : H \longrightarrow G$, C^0 gp. homo. $\Rightarrow C^\infty$

Pf. WLOG $H = \mathbb{R}$.
$\exp(nY) = \exp(Y)^n \rightsquigarrow$ reduce of nbd of e
same for $\varphi \rightsquigarrow$ 'same' $\Rightarrow C^\infty$.

§ Adjoint representation.

$$g \in G \overset{\rho}{\curvearrowright} G \qquad \text{conjugation action.}$$

$$\rho(g) : G \longrightarrow G$$

$$g \cdot h = \rho(g)(h) = g h g^{-1}$$

$$\overset{\frac{d}{dx}}{\Longrightarrow} \quad \rho(g)_* : \underbrace{T_e G}_{\mathfrak{g}} \longrightarrow \underbrace{T_e G}_{\mathfrak{g}} \qquad (\because \rho(g)(e) = e)$$

$$\rightsquigarrow \quad \mathrm{Ad} : G \longrightarrow GL(\mathfrak{g}) \qquad \text{Adjoint repr.}$$

$$g \longmapsto \rho(g)_*$$

$$\overset{\frac{d}{dx}}{\Longrightarrow} \quad \underbrace{\mathrm{Ad}_*}_{\mathrm{ad}} : \underbrace{T_e G}_{\mathfrak{g}} \longrightarrow \underbrace{T_{id}(GL(\mathfrak{g}))}_{\mathrm{End}(\mathfrak{g})} \qquad (\because \mathrm{Ad}(e) = id)$$

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}) \qquad \text{adjoint repr.}$$

Prop: $\mathrm{ad}_X(Y) = [X, Y]$

Pf: View X, Y as left inv. v.f. on G

$$\mathrm{ad}_X Y = \frac{d}{dt}\Big|_{t=0} \underbrace{\mathrm{Ad}(e^{tX})}_{d\,\rho(e^{tX})} Y, \qquad \text{write } e^{tX} := \exp(tX) \text{ 1-parameter subgp.}$$

$$\rho(e^{tX}) = r_{e^{-tX}} \circ l_{e^{tX}} \quad \leftarrow \text{left \& right multi.}$$

$$(\mathrm{ad}_X Y)_e = \frac{d}{dt}\Big|_{t=0} (d\, r_{e^{-tX}}) \circ \underbrace{(d\, l_{e^{tX}})(Y_e)}_{Y_{e^{tX}}}$$

$$= (L_X Y)_e \underset{\text{for any mfd}}{=\!=} [X, Y]_e$$

Chapter 6. Hodge Theorem.

§ Riemannian metric

(M^n, g) i.e. $g \in \Gamma(M, \mathrm{Sym}^2 T^*_M)$

s.t. $\forall x \in M$, $g_x : T_xM \otimes T_xM \longrightarrow \mathbb{R}$

is an inner product (i.e. pos. def. symm. bilinear form).

Choose any local coord. $x^1, x^2, \dots, x^n$.

$\Rightarrow$ At x, $dx^1, dx^2, \dots, dx^n$ form base T^*_xM

$$\Rightarrow \quad g_x = \sum_{i,j} g_{ij}(x)\, dx^i \otimes dx^j \qquad (g_{ij}) = (g_{ji}) > 0$$

Example: $\mathbb{R}^n$, $g_{std} = \sum_i dx^i \otimes dx^i$, i.e. $g_{ij}(x) = \delta_{ij}$.

Example: $M \subseteq \mathbb{R}^N$ submfd., $g := g_{std}|_M$.

Nash embedding theorem: Every g on M arises this way.

Intrinsic Geometry: Geometry of M which depends only on g, but not on $M \subseteq \mathbb{R}^N$. Otherwise, call extrinsic geometry.

Exercise: $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $M = \mathrm{Graph}(f) \subseteq \mathbb{R}^{n+1}$

i.e. $M = \{(x^1, \dots, x^{n+1}) : x^{n+1} = f(x^1, \dots, x^n)\}$. What is g on M?

Remark: Locally, we can always find coord. s.t. $g_{ij}(x) = \delta_{ij} + O(|x|^2)$ (Exercise). If we have $g_{ij}(x) = \delta_{ij} + O(|x|^3)$ then we have $(M^n, g) \cong (\mathbb{R}^n, g_{std})/\Gamma$. Those 2^{nd} order terms are called the Riem. curvature of M.

• g_x on $V = T_xM$
$\rightsquigarrow g_x$ on $\Lambda^k V^*$ for any k.

in such a way that if $dx^1, \dots, dx^n$ is an orthonormal base of V^*, then $\{dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}\}_{i_1 < i_2 < \dots < i_k}$ is o.n. base for $\Lambda^k V^*$

Exercise: Show that this is well-defined.

Eg: $u = u^i \frac{\partial}{\partial x^i}$, $v = v^i \frac{\partial}{\partial x^i} \in V$ (at p)

$g(u,v) = \sum_{i,j} g_{ij} u^i v^j$ (not nec. o.n.)

$\varphi = \varphi_{ij}\, dx^i \wedge dx^j$, $\eta = \eta_{ij}\, dx^i \wedge dx^j \in \Lambda^2 V^*$

$g(\varphi, \eta) = \sum g^{il} g^{jk} \varphi_{ij} \eta_{lk}$

where $g^{ij} g_{jk} = \delta^i_k$, i.e. $(g^{ij}) = (g_{ij})^{-1}$.

Exercise: The 2 unit length elts in $\Lambda^n V^* \simeq \mathbb{R}$ are

$$\nu = \pm \sqrt{\det(g_{ij})}\, dx^1 \wedge \dots \wedge dx^n.$$

If M has an orientation, then ν will be chosen to be in the same ori. and called the (Riemannian) volume form.

• Hodge star operator: $* : \Lambda^k V^* \longrightarrow \Lambda^{n-k} V^*$

$$\Lambda^k V^* \xrightarrow[\text{can.}]{\simeq} \underbrace{\Lambda^{n-k} V}_{V^*} \otimes \underbrace{\Lambda^n V^*}_{\mathbb{R}}$$

via: g $\qquad \nu^{-1}$

Exercise: (i) $*^2 = ?$

(ii) $\langle \varphi, \eta \rangle * 1 = \varphi \wedge * \eta = \eta \wedge * \varphi$

(iii) $*(dx^1 \wedge dx^2) = ?$ if dx^i's : o.n. oriented base.

Exercise: Given any $v \in V$, let

$$v^\flat := v \lrcorner g = g(v, -) \in V^*$$

(i.e. If $v = v^i \frac{\partial}{\partial x^i}$, then $v^\flat = g_{ij} v^i dx^j$)

Show that for any $\varphi \in \Lambda^k V^*$, $\eta \in \Lambda^{k+1} V^*$

we have $\langle \varphi, v \lrcorner \eta \rangle = \langle v^\flat \wedge \varphi, \eta \rangle$.

i.e. $v^\flat \wedge$ is the adjoint of $v \lrcorner$.

- Inner product on $\Omega^k(M)$: (need M oriented)

$$\Omega^k \times \Omega^k \xrightarrow{g} C^\infty(M) \xrightarrow{\int (-)\nu} \mathbb{R}$$

$$\langle\langle \varphi, \eta \rangle\rangle := \int_M g_x(\varphi, \eta)\, \nu(x)$$

$$\overset{\text{Ex.}}{=} \int \varphi \wedge * \eta = \int \eta \wedge * \varphi$$

Define: $d^* = - * d * : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$

and $\Delta = d d^* + d^* d : \Omega^k(M) \circlearrowleft$ Laplacian

$= (d + d^*)^2$ $\qquad \because (d^*)^2 = 0.$

Exercise: (i) Suppose M compact. Show that

$$\int \langle \varphi, d^* \eta \rangle \nu = \int \langle d\varphi, \eta \rangle \nu .$$

i.e. d^* is the formal adjoint to d.

(ii) $\langle\langle \varphi, \Delta \varphi \rangle\rangle = \| d\varphi \|^2 + \| d^* \varphi \|^2$

In particular, Δ is a non-negative operator with $\operatorname{Ker} \Delta = \operatorname{Ker} d \cap \operatorname{Ker} d^*$

Note: $d(\varphi \wedge \eta) = (d\varphi) \wedge \eta \pm \varphi \wedge (d\eta)$

But no such simple formula for d^*.

Lemma: $d\Delta = \Delta d$, $d^*\Delta = \Delta d^*$

Pf: $\Delta(d\varphi) = d^*\underbrace{d(d}_{0}\varphi) + dd^*(d\varphi) = dd^*d\varphi$

$= d(d^*d\varphi) + \underbrace{d(d}_{0}d^*\varphi) = d(\Delta\varphi)$

Given $f \in C^\infty(M) = \Omega^0(M)$, we write

$\nabla f = (df)^\# = \sum g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$ gradient vector field.

Exercise: $\Delta(fg) = f\Delta g + 2\langle \nabla f, \nabla g\rangle + g\Delta f$

Exercise: Show that in local coordinate

$\Delta f = \frac{-1}{\sqrt{g}} g^{ij} \partial_j(\sqrt{g}\,\partial_i f)$ where $\sqrt{g} := \sqrt{\det(g_{ij})}$

Exercise: Suppose $\dim M = 2$ and $g' = e^u g$ for some fu. $u: M \to \mathbb{R}$, show that $\Delta f = 0$ iff. $\Delta' f = 0$.

§ Linear Algebra aspects of Hodge Theory

Given a FINITE DIM. complex,

$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \longrightarrow 0$, $d^2 = 0$.

$H^k \triangleq \left.\frac{\operatorname{Ker}(d)}{\operatorname{Im}(d)}\right|_{\Omega^k}$ (measure failure of exactness)

Choose ANY metric $\langle\ \rangle_k$ on Ω^k.

Let $d^*: \Omega^k \longrightarrow \Omega^{k-1}$ be the adj. of d,

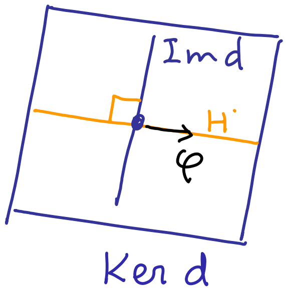
i.e. $\langle \varphi, d^*\eta\rangle = \langle d\varphi, \eta\rangle$

Define $\Delta = dd^* + d^*d = (d+d^*)^2 : \Omega^k \circlearrowleft$

- $\Delta = \Delta^* \geq 0$
- $\operatorname{Ker}\Delta = \operatorname{Ker} d \cap \operatorname{Ker} d^*$ hamonic elts.

Lemma: $H^k \simeq \operatorname{Ker} \Delta|_{\Omega^k}$.

$[\because \varphi \perp \operatorname{Im} d \iff d^*\varphi = 0]$



Exercise (i) Assume $(\Omega^\cdot, d)$ is a (comm.) diff. graded alg. (D.G.A).

i.e. $d(\varphi \wedge \eta) = (d\varphi)\wedge\eta + (-1)^{\deg\varphi}\varphi\wedge d\eta$. Then $\wedge$ decends to $H^\cdot$ & makes it a graded alg.

$$H^p \times H^q \xrightarrow{\wedge} H^{p+q}$$

(ii) If, moreover, $\exists$ linear map $\int : \Omega^n \longrightarrow \mathbb{R}$ satisfying $\int d\varphi = 0 \quad \forall\, \varphi \in \Omega^{n-1}$.

Then it gives a pairing,

$$H^k \times H^{n-k} \xrightarrow{\wedge} H^n \xrightarrow{\int} \mathbb{R}$$

(iii) Also assume $\Omega^k \times \Omega^{n-k} \xrightarrow{\wedge} \Omega^n \xrightarrow{\int} \mathbb{R}$ perfect pairing

$\rightsquigarrow$ Define star operator,

$$* : \Omega^k \xrightarrow[\int]{\text{via}} (\Omega^{n-k})^* \xrightarrow[\langle\ \rangle_{n-k}]{\text{via}} \Omega^{n-k}$$

i.e. $\int \varphi\wedge\beta = \langle *\varphi, \beta\rangle_{n-k}$ $\quad \forall\, \varphi \in \Omega^k$, $\forall\, \beta \in \Omega^{n-k}$

$= \langle \varphi, *\beta\rangle_k$

If $\langle\ \rangle_k$ on Ω^k is chosen s.t.

$\Omega^k \xrightarrow[\int]{\text{via}} (\Omega^{n-k})^*$ is an isometry.

Show that $*^2 = \pm 1$; $d^* = -*d*$; $*\Delta = \Delta *$.

As a result, $H^k \times H^{n-k} \xrightarrow{\wedge} H^n \xrightarrow{\int} \mathbb{R}$ is also a perfect pairing and (Poincaré duality)

$* : H^k \xrightarrow{\cong} H^{n-k}$ isometry.

Remark: $(\Omega^\bullet, \wedge, d)$ does contain more info. than $(H^\bullet, \wedge)$. (Massey product, min. model, formality) see Bott-Tu.

• Eigenspace decomposition.

Since $\Delta = \Delta^* : \Omega^k \circlearrowleft$. Consider the eigenspace decomposition, $\Omega^k = \bigoplus_{\lambda \geq 0} \Omega^k_\lambda$ w/ $\varphi \in \Omega^k_\lambda$ iff $\Delta\varphi = \lambda\varphi$.

Lemma: $d : \Omega^k_\lambda \longrightarrow \Omega^{k+1}_\lambda$ & $d^* : \Omega^k_\lambda \to \Omega^{k-1}_\lambda$.

Pf: $d\Delta = \Delta d$ & $d^*\Delta = \Delta d^*$ ∎

In particular, for any λ, we have a cpx.,

$$0 \to \Omega^0_\lambda \xrightarrow{d} \Omega^1_\lambda \xrightarrow{d} \cdots\cdots \to \Omega^n_\lambda \to 0.$$

Theorem: (i) $d = 0$ on $\Omega^\bullet_{\lambda=0}$.

(ii) When $\lambda \neq 0$, $(\Omega^\bullet_\lambda, d)$ is exact. ($\operatorname{Im} d = \operatorname{Ker} d$)

Pf: (i) is obvious. For (ii) $\operatorname{Im} d \subseteq \operatorname{Ker} d$ ✓.

Suppose $\varphi \in \Omega^k_\lambda \cap \operatorname{Ker} d$

$$\Delta\varphi = dd^*\varphi + d^*\underbrace{d\varphi}_{0} = \lambda\varphi$$

i.e. $\varphi = d\left(\frac{1}{\lambda} d^*\varphi\right)$ $(\because \lambda \neq 0)$

In particular, $\sum_{i=0}^{n} (-1)^i \dim \Omega^i_\lambda = 0 \quad \forall \lambda \neq 0$

$$\chi := \overbrace{\sum_i (-1)^i \dim H^i}^{\text{"dim"} H^\bullet}$$

$$= \sum_i (-1)^i \dim \underbrace{\Omega_0^i}_{} + \sum_{\lambda \neq 0} \underbrace{\left(\sum_i (-1)^i \dim \Omega_\lambda^i \right)}_{0}$$

$$= \sum_i (-1)^i \dim \Omega^i$$

$$=: \text{"dim"} \, \Omega^\bullet$$

Again $\forall t$,

$$\chi = \sum_i (-1)^i \dim \Omega_0^i + \sum_{\lambda \neq 0} e^{-t\lambda} \underbrace{\left(\sum_i (-1)^i \dim \Omega_\lambda^i \right)}_{0}$$

$$= \sum_i (-1)^i \underbrace{\left(\sum_\lambda e^{-t\lambda} \dim \Omega_\lambda^i \right)}_{\mathrm{Tr}\left(e^{-t\Delta} : \Omega^i \circlearrowleft\right)}$$

Heat operator $e^{-t\Delta} : \Omega \longrightarrow \Omega$

Exercise: (i) $e^{-(t_1+t_2)\Delta} = e^{-t_1\Delta} \circ e^{-t_2\Delta}$

Given any $\varphi_0 \in \Omega$, define $\varphi_t := e^{-t\Delta}\varphi_0$

Show that (ii) $\left(\frac{d}{dt} + \Delta\right)\varphi_t = 0$.

(iii) If $d\varphi_0 = 0$, then $d\varphi_t = 0$ and

$$[\varphi_t] = [\varphi_0] \in H \quad \forall t$$

(iv) Show that $\varphi_\infty := \lim_{t \to \infty} \varphi_t$ exists, $\Delta\varphi_\infty = 0$ and it is the only elt. in $[\varphi_0]$ which is in $\mathrm{Ker}\,\Delta$, i.e. harmonic.

(v) Show that the vector field on the vector space Ω given by $\varphi \mapsto \Delta\varphi$ is the gradient vector field for the functional

$$\underbrace{\tfrac{1}{2}|(d+d^*)\varphi|^2}_{\tfrac{1}{2}|d\varphi|^2 + \tfrac{1}{2}|d^*\varphi|^2} : \Omega \longrightarrow \mathbb{R}$$

Define $\mathcal{H} = \lim_{t\to\infty} e^{-t\Delta}$

$$\Delta = \begin{pmatrix} 0 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix} \Rightarrow e^{-t\Delta} = \begin{pmatrix} \overbrace{e^0}^{1} & & \\ & e^{-t\lambda_2} & \\ & & \ddots \end{pmatrix} \xrightarrow{t\nearrow\infty} \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots \end{pmatrix}$$ if $\lambda_2 > 0$.

i.e. $\mathcal{H}: \Omega \to \Omega$ is orthogonal proj. to harmonic elts.

Define Green operator: $G: \Omega \longrightarrow \Omega$

$$G = \begin{pmatrix} 0 & & \\ & \lambda_2^{-1} & \\ & & \ddots \end{pmatrix}$$ i.e. inverse on Δ on $(\operatorname{Ker}\Delta)^\perp$.

We then have $I = \mathcal{H} + G\Delta = \mathcal{H} + \Delta G$.

i.e. $\varphi = \underbrace{\underbrace{\mathcal{H}\varphi}_{\operatorname{Ker}\Delta} + d(d^*G\varphi)}_{\ker d} + d^*(dG\varphi)$

Also this is orthogonal decomposition.

We need to apply above studies to the ∞ dim. setting of diff. forms on M.
In fact, they can applied to other situations, e.g. d twisted w/ a flat connection, $\bar\partial$ operators.

§ Hodge theorem.

In our ∞ dim situation $\Omega^\bullet(M)$ we need ellipticity of Δ to ensure that

- $\dim \Omega^\bullet_\lambda < \infty$ $\forall \lambda$,
- λ_k grows very fact in k s.t. $\operatorname{Tr} e^{-t\Delta}$ is well-defined.

§ Proof of Hodge Theorem

Assume (A) and (B) (Pf ~ refer to Lawson's 'Spin Geometry.)

(A) Theorem (Regularity) $\alpha \in \Omega^p(M)$

if $\Delta\omega = \alpha$ weakly,

then $\Delta\omega = \alpha$ (strongly).

If $\Delta\omega = \alpha$, then

$$\underset{\langle \alpha, \beta \rangle}{\langle \Delta\omega, \beta \rangle} \xlongequal[\text{'by part'}]{} \langle \omega, \Delta\beta \rangle \quad \forall \beta$$

"weakly" means a bdd cts lin. $\ell_\omega : \overline{\Omega^p} \longrightarrow \mathbb{C}$

s.t. $\ell_\omega(\Delta\beta) = \langle \alpha, \beta \rangle \quad \forall \beta$

(B) Theorem: $\alpha_1, \alpha_2, \cdots \in \Omega^p(M)$ w/ $\|\alpha_i\|_{(2)}, \|\Delta\alpha_i\|_{(2)} \leq C$

$\Rightarrow \exists$ Cauchy subseq.

Cor: (1st eigenvalue estimate)

$$\beta \perp \mathrm{Ker}\,\Delta \Rightarrow \|\beta\|_{(2)} \leq C\|\Delta\beta\|_{(2)}$$

Pf: If NOT, $\exists$ such β_i's $\|\beta_i\| = 1$ & $\|\Delta\beta_i\|_{(2)} \xrightarrow{i\to\infty} 0$

Thm $\Rightarrow$ Cauchy seq. (up to subseq.)

$\Rightarrow \exists$ bdd linear $\ell : \Omega^p \longrightarrow \mathbb{C}$ w/ $\ell(\psi) := \lim_{i\to\infty} \langle \beta_i, \psi \rangle$

$\ell(\Delta\varphi) = 0 \quad (\because \|\Delta\beta_i\| \to 0)$

i.e. weak sol^{n} to $\Delta\beta = 0 \xRightarrow[\text{Thm.}]{\text{reg.}} \Delta\beta = 0 \ \exists \beta$

i.e. $\beta_i \to \beta \neq 0$ $(\because \|\beta_i\| = 1)$. But $\beta \perp \mathrm{Ker}\,\Delta$ ↯.

Proof of Hodge Decomposition $\Omega^p = \mathrm{Im}\,\Delta \overset{\perp}{\oplus} \mathrm{Ker}\,\Delta$.

$[\,(\mathrm{Ker}\,\Delta)^\perp \subset \mathrm{Im}\,\Delta\,]$ $\quad$ ($[\,\mathrm{Im}\,\Delta \subset (\mathrm{Ker}\,\Delta)^\perp\,]$ obvious)

$\alpha \in (\mathrm{Ker}\,\Delta)^\perp \rightsquigarrow \ell : (\mathrm{Im}\,\Delta) \longrightarrow \mathbb{C}$

$\ell(\Delta\varphi) := \langle \alpha, \varphi \rangle$ $\quad$ (well-def^{d} since $\langle \alpha, \Delta(\omega) \rangle = 0$)

$$|\ell(\Delta\varphi)| \leqslant \|\alpha\|\,\|\varphi\| \quad \left(\begin{array}{l}\text{replace } \varphi \\ \text{by one w/ } \varphi\in(\operatorname{Ker}\Delta)^{\perp}\end{array}\right)$$

$$\overset{\text{Thm}}{\leqslant} C\|\alpha\|\,\|\Delta\varphi\|$$

i.e. ℓ bdd linear fcl

$\xRightarrow{\text{Hahn-Banach}}$ extend $\ell : \Omega^p \longrightarrow \mathbb{C}$

i.e. weak sol$^{\underline{n}}$ to $\Delta\omega = \alpha$

$\xRightarrow{\text{Regularity}}$ $\exists\, \omega\in\Omega^p$ s.t. $\Delta\omega = \alpha$

i.e. $\alpha \in \operatorname{Im}\Omega^p$. QED

Application of the Hodge thm.,

(1) $\dim H^k(M) < \infty$ if M cpt ori. mfd

(2) Poincaré duality, $H^k(M) \times H^{n-k}(M) \xrightarrow{\wedge} H^n(M) \xrightarrow{\int} \mathbb{R}$

is perfect pairing. $* : H^k(M) \xrightarrow{\cong} H^{n-k}(M)$

(3) Bochner method (use the can. repr.).

eg. $Rc > 0 \Rightarrow b_1 = 0$.

(4) Hodge (p,q)-decomposition, $H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M)$,

$H^{p,q} = \overline{H^{q,p}}$. for compact Kähler mfds.

(5) Hard Lefschetz $sl(2,\mathbb{R})$-action on $H^{\cdot}(M,\mathbb{R})$

for compact Kähler mfd.